Wheel–Soil Interaction Model for Rover Simulation and Analysis Using Elastoplasticity Theory

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Abstract—A novel approach is proposed for the modeling of rigid-wheel and soft-soil interaction to efficiently compute normal and shear stress distributions in the contact area. The authors propose a velocity field in the vicinity of the contact area based on the physical nature of the problem. Thereupon, the incremental changes to the stress field are computed by resorting to elastoplasticity theory and an appropriate already existing constitutive relation for soil. The proposed approach leads to results that agree well with those obtained using well-established terramechanics models, while addressing some of their shortcomings. In addition, the proposed approach uses generalized velocities of the wheel as inputs, which makes it compatible with dynamic models of multibody systems. The dynamic slip–sinkage behavior of the wheel and the semieliptical shape of the normal stress distribution under the wheel are natural outcomes of the proposed model. Experimental investigations under various ranges of wheel slippage show good agreement with the data available in the literature.

Index Terms—Contact modeling, dynamics, multibody systems, wheeled robots, wheel–soil interaction.

I. INTRODUCTION

MOBILE robotic systems represent key elements for future planetary exploration as well as earthly applications. Such robots have to operate on different types of unstructured terrain, among which soft deformable soil is of particular interest. In order to investigate the effect of deformable soil on the performance of rovers, appropriate models are required to represent the interaction between wheel and terrain. In this context, soil reactions are required in response to the wheel movement.

The Bekker model [1], [2] and the extension made by Wong and Reece [3] are two semiempirical terramechanics models that are widely used, as they have been experimentally validated. The latter is referred to as the Wong–Reece (WR) model in this paper. These two models have a broad range of application in characterizing vehicles on soft terrain. Both models have significant application in mobile robotics as well. For example, in the AESCO Soft Soil Tyre Model (AS²TM) [4], the Bekker model is used. Furthermore, Shibiy et al. [5], Iagnemma and Dubowsky [6], Ishigami et al. [7], [8], and Hutangkabodee et al. [9] used the WR model in their rigid wheel–soil interaction studies. In addition, Wong and Asnani [10] compared the performance of several wheels of lunar vehicles by means of the NWVPM software package [11], in which normal (radial) stress distribution under the wheel is obtained using the Bekker model. A simplified version of the WR model was used by Iagnemma et al. [12] to identify cohesion and internal friction angle of soil for real-time applications to rovers operating on soft soil. Terrain parameter identification was also done by Ray [13], using the WR model. This WR model was also used by Ojeda et al. [14] for wheel slip detection and positioning error compensation.

The motivation of this paper is the need of a model for wheel–soil interaction that is compatible with multibody dynamics models and simulation environments. In this context, we need to determine soil reactions—forces and moments—using the model and the state of the wheel. In addition, the computational cost should be modest. The Bekker and WR models have these features; however, as they were not developed for the aforementioned conditions, they cannot represent some physical phenomena, which are explained below. Such models can still be used in multibody simulation by considering their limitations and using the necessary modifications. A possible implementation of the Bekker and WR models in a multibody dynamics simulation environment was reported by Azimi et al. [15]. In addition, an extension of the model that includes operation on rough deformable terrain with compaction and hardening of soil (multipass effect) was developed by Azimi et al. [16]. Other examples of the application of these models in multibody simulation studies can be found in [17]–[21].

In addition to the aforementioned models, other models, based on continuum mechanics, can be employed. In this regard, soft soil is modeled as a continuum, in which wheel–soil contact can be analyzed by considering an appropriate constitutive relation for soil and using detailed finite-element discretization to calculate stress distribution and soil deformation in the contact area, as recently reported in [22]–[24]. In yet another class of methods, dry soil is modeled as cohesionless granular material, with wheel–soil contact analyzed with the discrete element method (DEM) [11]. Today, one of the issues with DEM in wheel–soil interaction modeling is the need to consider a large number of particles, which results in extremely high computing time, even with supercomputers [11]. For wheel–soil interaction, finite-element analysis (FEA) is computationally less demanding than DEM. However, FEA is still inappropriate for
a multibody dynamics simulation environment, because of its high computational cost.

An efficient novel approach, based on elastoplasticity theory, is introduced in this paper for wheel–soil interaction. This approach will be shown to extend the applications domain of the Bekker and WR models, while being compatible with dynamics formulations and multibody simulation environments.\footnote{For example, in this paper, our implementations are based on Vortex, http://www.vxsim.com/} In this context, instead of resorting to FEA to find soil reactions on the rigid wheel, an assumed velocity field in the contact region is used. A rather simple still plausible velocity field is assumed that can lead to acceptable results, which are comparable with those obtained with the Bekker and WR models and experimental data. Another element of the proposed model is an elastoplastic constitutive relation to capture soil response. A Drucker–Prager constitutive relation with capability hardening [25], [26] is used, in this paper, to capture the plastic behavior of soil. This relation has been used in simulation of wheel–soil interaction using Abaqus/Explicit; examples of such studies are [22]–[24]. The proposed approach, moreover, is modular in that alternative forms of velocity fields and elastoplastic constitutive relations can also be employed in order to extend the range of applicability of the model.

As explained in Section IV-E, the computation time associated with the proposed approach is much shorter than that required by FEA. In addition, the proposed model is perfectly data-parallel in nature and, therefore, readily parallelizable, which makes the implementation well suited for execution on modern multicore architectures. Therefore, the proposed model can be used for fast and even real-time simulation of rovers on soft soil (with rover speed of a few centimeters per second).

Our contribution in this paper lies in proposing and developing a new framework for wheel–soil interaction with the features listed as follows.

1) The framework is algorithmic and modular; its implementation and step-by-step development are explained in Algorithms 1–3. In this framework, motion of soil particles near the contact area obeys an assumed velocity field. Using this velocity field, the strain increment tensor at any point on the contact region is determined based on the generalized velocities of the wheel. By means of an elastoplastic representation for soil, the stress distribution is obtained in the contact area without resorting to FEA. Additional simulation runs are included as well.

   Two algorithms are first developed: one is for the steady-state response and tuning the velocity-field parameters, and the other algorithm outlines the general dynamic motion implementation. Furthermore, a third algorithm is developed in which the elastic rebound of soil is computed. In addition, the slip–sinkage behavior is discussed by analyzing the motion of a wheel under a wide range of slip ratios. The results are also compared with experimental data available in the literature for various slip ratios. Additional simulation runs are included as well.

   The Bekker and WR models are briefly explained in Section II. The proposed approach is explained in detail in Section III, followed by simulation results and validation with reported experimental data in Sections IV and V. This paper ends with conclusions in Section VI.

II. SEMIEMPIRICAL TERRAMECHANICS MODELS

The aforementioned Bekker and WR models belong to the category of semiempirical terramechanics models, which are based on assuming a pressure–sinkage relation for the terrain. In these models, soil reaction is determined for a rigid wheel in planar motion and under steady-state conditions. In the Bekker model, the normal stress\footnote{In this paper, normal stress is meant normal to the wheel surface.} distribution \(\sigma_n\) in the contact area is determined by

\[
\sigma_n(\theta) = \left( \frac{k_c}{b} + k_d \right) \zeta^n(\theta) \tag{1}
\]

where \(k_c\) and \(k_d\) are pressure–sinkage parameters, \(n\) is the sinkage exponent, \(b\) is either the wheel width or the smaller dimension of the contact patch [11], and \(\zeta\) is the vertical sinkage at any point on the contact surface, as illustrated in Fig. 1. Moreover, \(k_c, k_d,\) and \(n\) are identified using the bevameter plate penetration test [11]. In this model, normal stress distribution under the wheel is determined based on the assumption that the stress is equal to that under a sinkage plate used in the bevameter test at the same depth [11]. In addition, the shear stress distribution \(\tau\)
is obtained by the expression
\[ \tau(\theta) = (c + \sigma_n(\theta) \tan \phi) \left[ 1 - \exp \left( -\frac{j_d(\theta)}{K_d} \right) \right] \tag{2} \]
where \( c \) and \( \phi \) are the cohesion stress and the angle of internal shearing resistance of the terrain, respectively, while \( K_d \) is referred to as the shear deformation modulus and has units of length and \( j_d \) is the shear displacement, with units of length as well. The latter has the form below for a driven wheel [3]:
\[ j_d(\theta) = R[(\theta_1 - \theta) - (1 - i_s)(\sin \theta_1 - \sin \theta)] \tag{3} \]
where \( \theta \) is defined in Fig. 1, \( \theta_1 \) is an angle indicating the initial contact with soil, \( R \) is the wheel radius, and \( i_s \) is the wheel slip ratio, which is defined as
\[ i_s = \frac{R\omega - v_x}{R\omega} \tag{4} \]
where \( \omega \) is the angular velocity of the wheel, and \( v_x \) is the horizontal component of velocity of the wheel center, for a wheel in planar motion.

Furthermore, in the WR model, the following relation is used to obtain the normal stress distribution in the contact area:
\[ \sigma_n(\theta) = \begin{cases} (k_c/b + k_o)c_n(\theta), & \text{if } \theta_M \leq \theta < \theta_1 \\ (k_c/b + k_o)[Ru_1(\theta)]^n, & \text{if } \theta_2 \leq \theta < \theta_M \end{cases} \tag{5} \]
where \( \theta_2 \) is the exit angle shown in Fig. 1, while \( u_1 \) is defined as
\[ u_1(\theta) = \cos \left( \theta_1 - \frac{\theta - \theta_2}{\theta_M - \theta_2}(\theta_1 - \theta_M) \right) - \cos \theta_1 \tag{6} \]
and angle \( \theta_M \) indicates the location of the point of maximum radial stress, namely,
\[ \theta_M = (c_1 + c_2 i_s)\theta_1 \tag{7} \]
in which \( c_1 \) and \( c_2 \) are dimensionless constants. Furthermore, the shear stress distribution is obtained based on (2) for this model as well. In this model, the normal stress distribution is similar to that in the Bekker model for \( \theta_M \leq \theta < \theta_1 \). In addition, \( \theta_2 = 0 \) is assumed in both models. When the normal and shear stress distributions around the wheel are known, using either the Bekker or the WR models, the soil reactions including motion resistance \( R_c \), traction force \( F_t \), resisting moment \( T_r \), and vertical load \( F_z \) can be obtained as follows:
\[ R_c = Rb \int_{\theta_2}^{\theta_1} \sigma_n(\theta) \sin \theta \, d\theta \tag{8} \]
\[ F_t = Rb \int_{\theta_2}^{\theta_1} \tau(\theta) \cos \theta \, d\theta \tag{9} \]
\[ T_r = R^2b \int_{\theta_2}^{\theta_1} \tau(\theta) \, d\theta \tag{10} \]
\[ F_z = Rb \int_{\theta_2}^{\theta_1} [\tau(\theta) \sin \theta + \sigma_n(\theta) \cos \theta] \, d\theta \tag{11} \]
and \( F_t - R_c \) is the drawbar pull of the wheel. It should be noted that the main point in the Bekker model is the assumption that the normal stress under the wheel is equal to the average pressure under a rigid plate pushed to the same depth. This assumption results in the maximum normal stress occurring at the bottom-dead-center. However, based on experimental evidence, the point of maximum normal stress shifts forward and may not be at the bottom-dead-center. This phenomenon is considered in the WR model.

Remark 1: It is noted that the Bekker model explained here is the one commonly used in the literature, for example, in [4] and [10], which is a modified version of the original Bekker model [1]. The differences are 1) the contribution of shear stress on the wheel center is considered; and 2) the shear displacement \( j_d \) is computed from (3).

A. Scope and Limitations of the Bekker and WR Models

It is noted that the Bekker and WR models were developed under certain assumptions, where a cylindrical wheel moves on a flat and horizontal soil surface under steady-state operations. In this context, given the vertical load on the wheel center and the slip ratio, these models make it possible to determine the wheel sinkage, drawbar pull, and driving torque. The scope and limitations of these models, coming from the above assumptions, are listed below. Our proposed model naturally addresses these limitations, as discussed below.

L1: If these models are directly used for dynamics simulation, the vertical component of the wheel-center velocity \( v_z \) has no contribution to the calculated soil reaction. Therefore, the soil reaction force in the \( z \)-direction \( (F_z) \) is only a function of sinkage \( z \) and slip ratio, which, in turn, means that the energy loss due to dynamic motion in the \( z \)-direction cannot be accounted for with the Bekker and WR models. Simulation of a wheel with either of these models results in an artificial oscillatory response in the vertical direction. A remedy to this problem is to add a nonlinear damping term, as discussed in [15] and [17]. In addition, abrupt changes of slip ratio, that can happen at low velocities, cause issues with the Bekker and WR models, which require special attention with the use of the slip ratio. On the other hand, in the model introduced in this paper, wheel velocity and angular velocity are directly used in the formulation (not the slip ratio). Therefore, the proposed model can handle dynamic situations including abrupt changes in the slip ratio.

L2: The slip–sinkage phenomenon is not included in the Bekker model. The slip–sinkage, however, is included to some extent in the WR model. There have been further attempts to address this issue; for example, in the work of Ding et al. [28], the exponent of the pressure–sinkage equation was assumed to be a linear function of slip ratio to match the experimental data, but no theoretical analysis was provided. With our proposed method, however, the slip–sinkage behavior is captured naturally.

Considering these limitations, modified Bekker and WR models can still be used in multibody dynamics simulation environments, as shown in [15] and [16]. In addition, there have been enhancements to these models; some of which are briefly discussed in Section II-B.
B. Recent Developments Pertinent to Rover Simulation

Some of the recent improvements to the Bekker and WR models related to rover simulation are listed as follows.\(^3\)

**Grousers**: The effect of grousers has been investigated with various methods. A simple and common approach, as in [30], is to use a larger wheel radius, i.e., original radius plus the grouser height. Another approach, however, is to consider individual bulldozing forces on the grousers [31]. In this paper, the grousers are not explicitly included; a larger-wheel-radius approach can be used to approximate the effects of the grousers.

**Computation of the exit angle \(\theta_2\)**: In the basic Bekker and WR models, a zero exit angle is assumed. Nonzero exit angle is usually determined by considering ground flexibility if the wheel is rigid (for example, in [4]). Another example for having nonzero exit angle is the approach presented in [7], in which a visually identified parameter \(\lambda\) was added to the model, from which the exit angle is directly determined. In our model, nonzero exit angle is the result of soil elastic rebound.

**Slip–sinkage behavior**: As briefly explained earlier, an example is the work of Ding et al. [28]. This behavior is captured in our model in a different way, without any particular assumption, as discussed in more detail in Section IV-C.

**Small rovers**: In [32], the pressure–sinkage equation of Bekker was extended to include wheel diameter with application to lightweight rovers with small wheels. In that approach, rigid cylinders are used in the pressure–sinkage experiments instead of flat plates.

**Dynamic conditions**: As briefly explained earlier, a common approach is to add a nonlinear damping term in the vertical response [15], [17]. There are also other modifications needed for these models to ensure that they can be used for the entire range of slip ratio [15], [7].

**Rough soil surface**: Examples are the works of Azimi et al. [16] and Ding et al. [20]. In these approaches, reasonable estimations of wheel sinkage and normal direction of the terrain are determined based on the contact geometry. Knowing the sinkage and normal terrain direction, modified Bekker and WR models are used to find soil reactions.

In all of the aforementioned approaches, normal stress distribution follows either the Bekker or the WR models.\(^4\) The exceptions are in [28] and [32]. In [28], the sinkage exponent \(n\) is a function of slip ratio to capture slip–sinkage, while in [32], pressure–sinkage relation is modified by including wheel radius in the equation in order to account for small wheels of lightweight rovers. As we will discuss in this paper, the slip–sinkage is included in our approach; also, we do not have any particular assumption for the weight of the rover or wheel diameter. However, the experimental results in Section V apply to a small rover.

\(^3\)Some aspects like wheel flexibility and lateral forces are not discussed here, as they do not lie within the scope of this paper. Further discussion on this topic is available in [29].

\(^4\)We have thus compared the stress distribution computed using our model with these basic models—see Section IV.

III. Soil Reaction Estimation Using Elastoplasticity

Assume that a rigid wheel, under planar motion, is in contact with homogeneous soft soil. In order to find soil reactions, the calculation or estimation of the normal and shear stress distributions around the wheel is required. Then, from (8)–(11), soil reactions will be calculated. A novel approach is developed in this section based on elastoplasticity theory to calculate \(\sigma_n\) and \(\tau\), in the contact area. The proposed approach comprises two stages. First, we explain how, for any point in the contact area, the change in the stress tensor \(\Delta\sigma\) caused by wheel motion is determined. Second, the stress field in the entire contact area is computed, using the algorithms introduced below.

A. Computation of the Stress Increment Tensor

Let us consider that at a point (in the soil) close to the wheel surface, the stress tensor \(\sigma\) and the current strain tensor \(\epsilon\) are known. In addition, the elastic \(\epsilon^e\) and plastic \(\epsilon^p\) parts of the strain tensor are also known. Given the state of the rigid wheel, we look for the \(\Delta\sigma\) developed in a small time-step \(\Delta t\). The main step is determining the strain increment tensor \(\Delta\epsilon\), which is done here by assuming a velocity field at the region near the contact area. The assumed velocity field and the determination of \(\Delta\epsilon\) are explained in this section.

Using elastoplasticity theory and a suitable constitutive relation for soil, the elastic and plastic parts of the strain increment, i.e., \(\Delta\epsilon^e\) and \(\Delta\epsilon^p\), are calculated based on an iterative procedure for finding a plastic multiplier [33]. Further details are available in Appendix B.

**Remark 2**: As mentioned by Khan and Huang [34], the constitutive equations for plastic deformation should be formulated in incremental form because of path dependence in plastic deformation.\(\square\)

In this paper, the additive decomposition of the strain tensor is adopted, which is expressed in incremental form as\(^5\)

\[
\Delta\epsilon = \Delta\epsilon^e + \Delta\epsilon^p.
\]  

(12)

As previously indicated, any elastoplastic constitutive relation can be used in our proposed approach to represent the soil response. In this paper, the Drucker–Prager model with cap hardening is used to express the plastic behavior of soil. In order to have a complete elastoplastic constitutive relation, an elastic model needs to be adopted as well, for which a linearly elastic relation has been used here. For the plastic behavior, yield surfaces and flow potentials should be described with any hardening or softening rule. These are explained in detail in [25] and [26]; a brief explanation is included in Appendix A.

The assumed velocity field is described now. By assuming that the friction coefficient between the wheel and the soil surfaces is higher than the internal friction coefficient of soil, any slippage happens between soil particles, not between wheel surface and soil. Therefore, the velocity of any soil particle in contact with

\(^5\)It is noted that in finite deformation, the decomposition of elastic and plastic parts are multiplicative [34]; however, as mentioned in [25], the additive decomposition still holds when elastic strain is infinitesimal and the strain rate is measured as the rate of deformation.
the wheel circumference is equal to the velocity of the corresponding point on the wheel surface. Under these conditions, the soil velocity field near the contact surface is assumed to have the form
\[ v_r(r, \theta) = (v_z \sin \theta - v_z \cos \theta) \exp \left[ -\frac{\alpha_1}{\varepsilon_{v_0} + \alpha_2} (r - R) \right] \] (13)
\[ v_\theta(r, \theta) = (v_z \cos \theta + v_z \sin \theta - r \omega) \exp \left[ -\alpha_3 (r - R) \right] \] (14)
where polar coordinates \( r \) and \( \theta \) are used to uniquely define the location of any point in the soil with respect to the wheel center, while \( v_z, v_z, \) and \( \omega \) are the generalized velocities of the wheel in planar motion, as shown in Fig. 2. In addition, \( (r - R) \) indicates the depth in the radial direction measured from the wheel surface, \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) being constant positive scalars, and \( \varepsilon_{v_0}^{-} \) the volumetric part of the plastic strain tensor. It is noted that the velocity field defined in (13) and (14) is valid for \( r \geq R \) and \( \theta_2 \leq \theta \leq \theta_1 \).

**Remark 3:** By assuming the presence of small narrow grousers on the wheel surface, slippage will happen only among soil particles. This assumption tallies with the presence of high friction between wheel surface and soil.

The assumed velocity field is in agreement with the boundary conditions of the problem, i.e., at \( r = R \), soil particle velocity is equal to that of the contact point on the wheel, while the velocity approaches zero as \( r \to \infty \). It is also noted that the motion of a soil particle under the wheel is a phenomenon of diffusion, as opposed to propagation; thus, the exponential decaying terms in the velocity field are consistent with this behavior. In addition, as discussed below, the velocity field is only used in the vicinity of the wheel surface, as the only use of the velocity field in our approach is to determine the velocity gradient in the contact area. The assumed velocity field is in agreement with the experimental observations reported in the literature [35], [36], where the motion of soil particles under the wheel is recorded using high-speed cameras; the velocity field is then visualized from postprocessing of the recorded images.

By using the aforementioned Drucker–Prager model for soil, the model predicts either elastic deformation or elastoplastic deformation with hardening or softening for soil, all depending on the state of stress. If the stress in the \( pq \) plane lies in the cap region \( (F_c) \), the soil shows compaction; otherwise dilation. In addition, \( p \) and \( q \) are the stress invariants defined in Appendix A.

Moreover, when the soil is loose, which means a small \( \varepsilon_{v_0}^p \), a surface-applied penetration, by a rigid wheel surface, for example, will mainly cause some compaction on the soil under the wheel, but will not cause considerable soil flow to the sides. However, the same penetration on the same soil but with a higher density could result in soil particle motion to the sides. This means that the soil with higher initial density will become less compacted when facing the same motion on its surface. This behavior is captured to some extent by including \( \varepsilon_{v_0}^p \) in the velocity field and the model parameter \( \alpha_2 \). More discussion on the assumed velocity field is included in Section IV-D.

**Remark 4:** The proposed velocity field is compatible with the aforementioned constitutive relation used for soil, as the hardening/softening pattern of soil is fully identified by \( \varepsilon_{v_0}^p \), which is a key parameter in the velocity field.

Then, the velocity gradient tensor at \( r = R^+ \) (on the external wheel periphery\(^6\)) is determined as
\[ \mathbf{G}_{r\theta}(R^+, \theta) = \begin{bmatrix} -\frac{\alpha_1}{\varepsilon_{v_0} + \alpha_2} v_r(R, \theta) & \omega \\ -\alpha_2 v_\theta(R, \theta) - \omega & 0 \end{bmatrix} \] (15)
where \( v_r(R, \theta) \) and \( v_\theta(R, \theta) \) are the radial and tangential velocity components at the wheel surface at location \( \theta \) and are obtained from (13) and (14), respectively.

Introducing the rotation matrix
\[ \mathbf{R} = \begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix} \] (16)
the velocity gradient tensor \( \mathbf{G}_{r\theta} \) can be transformed into the fixed-coordinate system \((xz)\) via a similarity transformation
\[ \mathbf{G}_{xz} = \mathbf{R} \mathbf{G}_{r\theta} \mathbf{R}^T \] (17)
where \( \mathbf{G}_{xz} \) is the velocity gradient tensor in the \( xz \) system. Then, the time derivative \( \dot{\varepsilon} \) of the infinitesimal strain tensor can be obtained as the symmetric part of the velocity gradient
\[ \dot{\varepsilon} = \frac{1}{2} (\mathbf{G}_{xz} + \mathbf{G}_{xz}^T). \] (18)
The strain increment is then obtained as
\[ \Delta \varepsilon = \dot{\varepsilon} \Delta t. \] (19)

### B. Determination of the Stress Tensor in the Contact Area

The procedure for obtaining soil reactions is explained in the two algorithms below. In Algorithm 1, it is assumed that the rigid wheel is in steady-state condition, similar to the Bekker and WR models, and moves with a nonzero angular velocity, while in Algorithm 2, the general dynamic motion is considered. Algorithm 1 is straightforward for implementation, compared with Algorithm 2, and is useful for parameter tuning and comparison.

\(^6\) \( R^+ \) indicates the value of \( r \) when \( r \) approaches \( R \) from ”the right,” as the function whose argument is \( r \) is not defined at \( R^+ \).
with the Bekker and WR models, as explained in Section IV-D. Algorithm 2, however, is applicable to the general dynamic motion of the wheel, as explained further below.

In Algorithm 1, the normal and shear stress distributions in the contact area are obtained by following the motion of a single point on the wheel periphery from its initial contact with soil ($\theta = \theta_1$) until separation ($\theta = \theta_2$ in Fig. 2). During this motion, at any location of this point identified by the angle $\theta$, shear and normal stresses are obtained. As the wheel moves in the steady-state condition, the stress values should not change in time at any contact angle $\theta$. Therefore, the calculated stress distributions, which are obtained by following the motion of the point mentioned above, represent the stress distributions under the wheel.

**Algorithm 1.**

**Assumptions:**
1) The rigid wheel operates in steady-state conditions. This means that the velocity component $v_z$ is zero. In addition, the stress field in the contact area should not change in time.
2) The precompaction level and the initial stress level of soil before contacting with the wheel are known. Therefore, the stress and strain tensors are known at the initial contact point $\theta = \theta_1$ (in Fig. 2).
3) A high friction coefficient exists between wheel surface and soil, as explained earlier and in Remark 3.

**Steps:**
1) At the initial contact point, $\theta = \theta_1$ in Fig. 1, initialize $\sigma$ and $\varepsilon$ from the initial compaction data of soil. Then, choose a small $\Delta t$ for integration, set $\Delta \theta = \omega \Delta t$, and go to Step 5.
2) Update $\theta = \theta_p - \Delta \theta$.
3) From (13)–(19), calculate $\Delta \varepsilon$ associated with the motion during $\Delta \theta$.
4) Using elastoplasticity theory, $\Delta \varepsilon^r$ and $\Delta \varepsilon^p$ are determined (see Appendix B), which leads to computation of $\sigma, \varepsilon^r,$ and $\varepsilon^p$ at the current location on the wheel (with angle $\theta$). Then, express the stress tensor in the $r$-$\theta$ directions to obtain $\sigma_{\alpha}(\theta)$ and $\tau(\theta)$.
5) Set $\theta_p = \theta$. If $\theta > 0$, go to Step 2; else, go to the next step.
6) If $\sigma_{\alpha}(\theta) \leq 0$, go to Step 8; else, soil particle is in the elastic rebound condition. Follow the steps in Algorithm 3 to find $\sigma$ and $\Delta \varepsilon^r$ $(\Delta \varepsilon^p = 0)$.
7) Express the stress tensor $\sigma$ in the $r$-$\theta$ directions to obtain $\sigma_{\alpha}(\theta)$ and $\tau(\theta)$. Then, go to Step 6.
8) Use (8)–(11) to obtain soil reactions.

**Remark 5:** By default, elastic rebound of soil is included in Algorithm 1 in steps 6 and 7. Omitting these steps means neglecting the elastic rebound of soil.

Algorithm 2 outlines the procedure involved in this novel approach, for the dynamic motion of a rigid wheel in planar motion. In this algorithm, an explicit integration scheme is used to obtain the updated normal and shear stress distributions under the wheel, based on the motion of the wheel and hardening state of the soil. Here, the stress values are obtained at some mesh points on the contact region of the wheel and soil (see Fig. 3).

**Algorithm 2.**

**Assumptions:**
1) The precompaction level and the initial stress level of soil before contacting with the wheel are known.
2) A high friction coefficient exists between wheel surface and soil, as explained earlier and in Remark 3.

**Steps:**
1) Determine some mesh points on the periphery of the wheel by creating a surface mesh on the wheel periphery. The nodes of this mesh are the mesh points used in this algorithm (see Fig. 3).
2) Determine the mesh points that are in contact with the terrain (referred to as active points). The values of stress and strain at active points that have not been in contact with the terrain in the previous time-step will be initialized from the initial stress and strain of soil.
3) For a given time-step and for each active point, the steps below are followed. It should be noted that for each active point, the current stress and strain tensors are known at time $t$.
   a) Define the loading condition: loading or unloading:
      i) Compute $v_z'$ as
         
         \[ v_z' = \mathbf{v}_{\text{mesh}} \cdot \mathbf{n}_{\text{terrain}} \]  (20)

         where $\mathbf{v}_{\text{mesh}}$ is the velocity vector of the active point, and $\mathbf{n}_{\text{terrain}}$ is the normal direction of the terrain.\(^7\)
      ii) If $v_z' > 0$, the active mesh point is in the rebound; go to Step 3(d). Otherwise, the point is in a loading condition; go to Step 3(b).
   b) $\Delta \varepsilon$ is calculated from (13)–(19).
   c) Using elastoplasticity theory, $\Delta \varepsilon^r$ and $\Delta \varepsilon^p$ are determined (see Appendix B), which leads to the updated stress tensor $\sigma(t + \Delta t)$ and updated strain

\(^7\)When operating on an irregular terrain, $\mathbf{n}_{\text{terrain}}$ can be computed by following the approach explained by Azimi et al. [16]. There, we have used a heightfield to represent the terrain surface, and we have computed a best-fit plane using least squares of the heightfield vertices in the so-called footprint of the wheel; $\mathbf{n}_{\text{terrain}}$ is the normal to that least squares plane.
\(^8\)For numerical stability reasons, in our implementation instead of zero, we compare $v_z'$ with a small positive scalar $\varepsilon'$ to start our elastic rebound phase. When $0 \leq v_z' \leq \varepsilon'$, soil will show rebound with softening.
Elastic rebound condition: Follow the steps in Algorithm 3 to find the updated stress tensor \( \sigma(t + \Delta t) \) and updated strain tensor \( \varepsilon(t + \Delta t) \), while \( \varepsilon^p \) remains unchanged.

4) Express the updated stress tensor of all active points in the \( r-\theta \) directions to obtain \( \sigma \) and \( \tau \). Integration of \( \sigma \) and \( \tau \) over the area of wheel surface covered by the active points will result in the reaction forces of soil.

5) Perform one increment of the multibody simulation with the reaction forces obtained in the previous step. Set \( t \leftarrow t + \Delta t \) and go to Step 2. \( \square \)

Remark 6: \( \Delta \varepsilon \) should be small enough to ensure the convergence of the numerical procedure involved in obtaining \( \Delta \varepsilon^e \) and \( \Delta \varepsilon^p \), in Step 3; the required time-step for the \( \Delta \varepsilon \) calculation is usually much smaller than the time-step of the multibody system simulation. In this case, Step 3 is conducted multiple times under smaller increments to ensure the convergence of the \( \Delta \varepsilon^e \) and \( \Delta \varepsilon^p \) calculations. \( \square \)

Remark 7: Having a large number of mesh points will increase the computation time; however, the algorithm is highly parallelizable as each mesh point can be treated independent of the other points in this algorithm. It should be noted that using a very small number of mesh points can cause non-negligible discretization error, which may result in noticeable oscillations. \( \square \)

C. Elastic Rebound Computation

In wheel and soil interaction, soil rebound happens when the wheel surface starts to separate from soil instead of pushing it. Therefore, in this stage, pressure between soil particles decreases, which causes reduction in both normal and shear stresses. In our model, during elastic rebound, normal and shear stresses decrease such that the hardening state of soil does not change (due to a purely elastic rebound assumption).

The assumed velocity field in this paper, which is expressed in (13) and (14), is valid for the region undergoing no rebound, because there it is assumed that the soil particle in the wheel surface follows the velocity of the wheel surface. However, if the soil particle at the wheel surface is in a rebound condition, then it cannot follow the wheel surface, because it cannot be pulled by the wheel, where negligible adhesion between wheel surface and soil is assumed. We can, however, assume that in the radial direction, the soil particle follows the motion of the wheel surface until the contact pressure between the wheel and soil reaches zero. After that, separation between wheel surface and the soil particle happens. During this rebound phase, the shear strain is adjusted in our algorithm such that the state of stress remains inside or on the yield surface (state of stress outside the yield surface is invalid), while the hardening state of soil does not change. To this end, Algorithm 3 is introduced for any point that is in the elastic rebound phase. It is noted that the condition for a soil particle to be in the elastic rebound phase was explained in Algorithms 1 and 2.

Algorithm 3.
Given the current stress tensor \( \sigma \), strain tensors \( \varepsilon^e \) and \( \varepsilon^p \), wheel geometry and state variables, and knowing that the soil particle is experiencing elastic rebound, compute updated stress and strain tensors after a small time-step \( \Delta t \).

Steps:

1) Compute \( \Delta \varepsilon \) from (19) and initialize the elastic and plastic parts of the strain increment tensor by

\[
\Delta \varepsilon^e_{\text{trial}} = \Delta \varepsilon \\
\Delta \varepsilon^p = 0.
\] (21a) (21b)

2) Introduce a trial stress state, which has the form below for a linearly elastic behavior

\[
\sigma^\text{trial} = \sigma + C : \Delta \varepsilon^e_{\text{trial}}
\] (22)

where \( C \) is the fourth-rank elastic stiffness tensor. Symbol \( " : " \) denotes double contraction, as needed between a fourth-rank tensor \( C \) and a second-rank tensor \( \varepsilon^e_{\text{trial}} \) [33], to produce a second-rank stress tensor.

3) Check the validity of \( \sigma^\text{trial} \). If it is inside or on the yield surface, go to step 5, as \( \sigma^\text{trial} \) is a valid stress tensor [33]; otherwise, go to the next step.

4) Modify \( \Delta \varepsilon^e_{\text{trial}} \): This is done by modifying the lower off-diagonal element of \( G_{r,\theta}(R^+, \theta) \) in (15), referred to as \( G_{r,\theta}(2,1) \), by the relation below

\[
G_{r,\theta}(2,1) = -\alpha_3 t g (R, \theta)(1 - \eta) - \omega
\] (23)

where \( \eta \) is a positive scalar. Here, we increase \( \eta \) stepwise from zero until the condition in step 3 is satisfied on \( \sigma^\text{trial} \). The updated \( \Delta \varepsilon^e_{\text{trial}} \) is computed from (13)–(19) and using (23). It is noted that, by using this modified strain increment tensor in the elastic rebound phase, the state of stress remains valid, while the hardening state of soil does not change. This is the necessary condition for a purely elastic rebound. After incrementing \( \eta \) and computing \( \Delta \varepsilon^e_{\text{trial}} \), go to step 2\(^9\).

5) Set \( \sigma = \sigma^\text{trial} \) and exit. \( \square \)

IV. Simulation Results and Further Analysis

To validate the proposed approach, our results are compared with those obtained using the Bekker and WR models. As mentioned above, the Drucker–Prager constitutive relation with cap hardening is used to model the plastic deformation of soil in this paper. Combined with a linear elasticity model, this relation leads to an elastoplastic constitutive relation for soil. However, the nature of the Bekker and WR models is different from this elastoplastic constitutive relation, and they have different sets of parameters. The parameters of the elastoplastic constitutive relation are listed in Tables I and II; some of these parameters are taken from [22].

Remark 8: It is noted that the approach proposed in this paper does not depend on a specific constitutive relation for soil or

\(^9\)Instead of increasing \( \eta \) stepwise, a more computationally efficient approach would be formulating a single-variable optimization problem, basically a line search, to find the appropriate \( \eta \).
TABLE I

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus, $E$ (Pa)</td>
<td>$3 \times 10^8$</td>
</tr>
<tr>
<td>Poisson’s ratio, $\nu$</td>
<td>0.32</td>
</tr>
<tr>
<td>Angle of friction, $\beta$ (deg)</td>
<td>41</td>
</tr>
<tr>
<td>Material cohesion, $d$ (Pa)</td>
<td>350</td>
</tr>
<tr>
<td>Cap eccentricity, $R_c$ (-)</td>
<td>0.15</td>
</tr>
<tr>
<td>Initial value for $C_\ell$ (-)</td>
<td>0.001</td>
</tr>
</tbody>
</table>

TABLE II

<table>
<thead>
<tr>
<th>$p_0$ (kPa)</th>
<th>$\lambda_0$</th>
<th>$p_0$ (kPa)</th>
<th>$\lambda_0$</th>
</tr>
</thead>
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<tr>
<td>1.5</td>
<td>0.014661</td>
<td>1.7</td>
<td>0.149679</td>
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<tr>
<td>3</td>
<td>0.028334</td>
<td>3.0</td>
<td>0.169028</td>
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<td>6</td>
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<td>0.185036</td>
</tr>
<tr>
<td>9</td>
<td>0.074619</td>
<td>9.0</td>
<td>0.208422</td>
</tr>
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<td>12</td>
<td>0.093572</td>
<td>12.0</td>
<td>0.220405</td>
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<tr>
<td>15</td>
<td>0.110262</td>
<td>15.0</td>
<td>0.248232</td>
</tr>
<tr>
<td>18</td>
<td>0.125906</td>
<td>18.0</td>
<td>0.269098</td>
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<tr>
<td>21</td>
<td>0.138069</td>
<td>21.0</td>
<td>0.280999</td>
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TABLE III

<table>
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<td>$\phi$ (deg)</td>
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</tr>
<tr>
<td>$c$ (Pa)</td>
<td>234</td>
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<tr>
<td>$k_\phi$ (N/m$^{(\phi-2)}$)</td>
<td>$4.104 \times 10^5$</td>
</tr>
<tr>
<td>$k_c$ (N/m$^{(\phi+1)}$)</td>
<td>0</td>
</tr>
<tr>
<td>$n$ (-)</td>
<td>0.8</td>
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<tr>
<td>$K_d$ (m)</td>
<td>0.013</td>
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<tr>
<td>$c_1$ (-)</td>
<td>0.18</td>
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<tr>
<td>$c_2$ (-)</td>
<td>0.32</td>
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</table>

TABLE IV

<table>
<thead>
<tr>
<th>Parameter</th>
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</thead>
<tbody>
<tr>
<td>$\alpha_1$ (m$^{-1}$)</td>
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</tr>
<tr>
<td>$\alpha_2$ (-)</td>
<td>0.03</td>
</tr>
<tr>
<td>$\alpha_3$ (m$^{-1}$)</td>
<td>35</td>
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</tbody>
</table>

Fig. 4. Comparison of the calculated drawbar pull for 20% slip ratio under the action of different vertical loads.

In the remainder of this section, an example of steady-state motion for a single wheel is analyzed, followed by nonsteady motion of the same wheel in the second example. Then, the slip-sinkage behavior is investigated. An explanation of the assumed velocity field parameters and general discussion are the topics of the last two sections.

A. Steady-State Motion

In this case, a rigid cylindrical wheel with 0.30 m diameter and 0.10 m width moves under steady-state conditions, which means a constant $v_x$ and $\omega$ and a zero $v_z$. $v_x$ is 0.12 m/s and $\omega$ is 1.0 rad/s, which results in 20% slip ratio. This example is repeated for different values of vertical load ranging from 39 to 206 N. Soil reactions are compared with the Bekker and WR models. The parameters of the proposed velocity field, which were tuned based on an approach explained in Section IV-D, are listed in Table IV. It should be mentioned that in the results displayed in Figs. 4–8, we have adopted a zero exit angle ($\theta_2 = 0$) by assuming that the elastic rebound of soil is negligible. As $\theta_2 = 0$ is also assumed in the Bekker and WR models, this assumption helps us do a fair comparison between the basic elements of our model and the aforementioned Bekker and WR models. After that, all the results are obtained with considering the elastic rebound of soil.

By comparing the results of the drawbar pull and the resistance force (see Figs. 4 and 5), the estimated resistance force...
Fig. 6. Comparison of the resulting sinkage for 20% slip ratio under the action of different vertical loads.

Fig. 7. Normal stress distribution for 20% slip ratio under $F_z = 165$ N.

Fig. 8. Shear stress distribution for 20% slip ratio under $F_z = 165$ N.

closely matches the ones resulting from the Bekker and WR models; however, the traction force is overestimated at low and underestimated at high sinkage values, when compared with these models. In addition, as shown in Fig. 6, the estimated wheel sinkage matches the Bekker and WR models relatively well, in various loadings and a fixed 20% slip ratio.

Furthermore, the normal and shear stress distributions are displayed in Figs. 7 and 8. The normal stress distribution is closer to the Bekker model, but shows that the position of maximum normal stress is shifted forward, which agrees with experimental evidence [11]. It is noteworthy that the normal stress distribution resulting from the proposed approach is in good agreement with these models. The shear stress, however, is overestimated at the area closer to the entry point and underestimated toward the bottom-dead-center. This can be related to a shortcoming of the adopted constitutive relation, which can represent the compaction process relatively well but overestimates the shear stress at small shear strains. This means that the proposed model with this constitutive relation can more accurately represent terrains with small $K_d$. It should, however, be noted that the Bekker and WR models provide an estimation for the stress distribution in the contact area; further experimental data, for stress distribution, are needed to comment more specifically on the validity of these results.

As shown in Figs. 7 and 8, the stress values become zero for negative contact angles ($\theta < 0$). It should be noted that in steady-state operations ($v_z = 0$), the radial component of the velocity vector on the wheel periphery at angle $\theta$, $v_r (R, \theta)$ becomes negative for $\theta < 0$, and points toward the wheel center. For this case, if we assume that the soil shows negligible elastic rebound, there will be no contact when $v_r (R, \theta) < 0$, as the wheel can only push the soil. Therefore, assuming zero elastic rebound when $v_z = 0$ results in $\theta_2 = 0$, and a discontinuity in the stress distribution.

This example was analyzed again while considering the rebound of soil due to its elasticity and using a relatively small Young’s modulus for soil (see Young’s modulus in Table I). Normal and shear stress distributions are shown in Fig. 9. As a relatively small Young’s modulus is used for soil, the rear region ($\theta_2 < \theta < 0$), corresponding to the elastic rebound, is noticeable. Increasing the Young’s modulus results in smaller $\theta_2$. As expected, the stress distributions show no discontinuity in this case.

Remark 9: As mentioned above, the nonzero exit angle in this model is the result of the elastic soil rebound. In practice, soil flow and grousers affect the exit angle, especially when moving on sand with relatively high slip ratio. However, when adopting a continuum model with an elastoplastic soil representation, soil (and wheel) elasticity determines the exit angle, which is an inherent limitation of elastoplastic soil representation. It should be mentioned that the common practice in the literature is to either assume the exit angle as a function of the entrance angle, as in [7], or use soil and wheel elasticity, as in [4].

B. Nonsteady Motion and Variable Slippage

In this section, we analyze a simple but illustrative example in order to demonstrate the behavior of our model in nonsteady
operations and under a wide range of wheel-slipage conditions. In this example, we investigate the planar motion of a rigid wheel on soil, in which $v_x$ and $\omega$ are controlled to achieve certain values for the wheel slip ratio with $\omega$ measured positive cw and $v_x$ positive to the right (see Fig. 1). In the $z$-direction, the wheel is free to move under gravity. Mass, radius, and width of the wheel are 16 kg, 0.15 m, and 0.10 m, respectively. Soil parameters are as displayed in Tables I–IV, except that $c_2 = 0.6$ is used to better visualize the slip dependence in the WR model with a large value for $c_2$. It is noted that $c_1 + c_2$ cannot be larger than 1.0; moreover, the higher the $c_2$ value, the higher the slip–sinkage effect. In addition, $k_\phi = 5.5 \times 10^5$ N/m$^{(n+2)}$ is used.

The wheel is dropped with a zero initial velocity and an initial sinkage of 1 mm. Algorithm 2 is used to obtain the reaction forces applied on the wheel; the elastic rebound of soil is included, which leads to a nonzero exit angle for the proposed model. From $t = 0$ to $t = 1$ s, the wheel moves in the $z$-direction only, due to gravity, and causes plastic deformation in soil. The wheel is then commanded to move forward with the velocity profiles shown in Fig. 10. With these velocity profiles, the slip ratio grows incrementally from 0% to 30%.

Again, our results are compared with those obtained based on the Bekker and WR models. However, directly using the Bekker or WR model in this example leads to an unrealistic oscillatory response in soil reactions and wheel sinkage, because energy dissipation in the $z$-direction is not considered in those models, as briefly explained in Section II-A. Therefore, we used the modified version of these models, as proposed by Azimi et al. [15]. The results obtained with our proposed approach and with the modified Bekker and WR models are displayed in Figs. 11–13.

As can be seen from Fig. 11, using our model, wheel sinkage increases with slip ratio, which agrees qualitatively well with the experimental observations, known as slip–sinkage. In the Bekker model, however, this feature is not captured, while in the WR model, it is captured to some extent, if a relatively high value for $c_2$, which defines the slip dependence in the WR model, is used. In addition, at the initial phase, when the wheel is dropped, it sinks with a negligible rebound when the proposed model is used, due to plastic deformation of soil. The wheel moves up when it starts its motion with zero slip ratio, which agrees with experimental observations.

In Fig. 12, the traction and resistance forces predicted by our model exhibit some oscillations from their nominal value. This is caused by the discretization error associated with the use of mesh points. Increasing the number of the mesh points results in smaller oscillation amplitudes in the reaction forces.

C. Analyzing the Slip–Sinkage Phenomenon

According to the WR model [3], the effect of wheel slip on normal stress distribution under a rigid wheel is at the position of maximum radial stress ($\theta_M$), but the stress distribution from soil surface to $\theta_M$ remains independent of wheel slip and follows Bekker’s pressure–sinkage relation. However, the results of our model suggest that the wheel slip affects the stress distribution in the entire contact area.
By increasing wheel slip, according to our model, soil particles at the contact area experience a higher shear deformation (in the direction tangent to the wheel surface) while receiving a lower push from the wheel in the direction normal to the wheel surface. Based on our assumed velocity field, and the Drucker–Prager constitutive relation adopted for soil, this combination leads to a strain increment tensor $\Delta \varepsilon$ with a smaller volumetric plastic part. As a result, a lower soil hardening is predicted with our model when wheel slip increases. The pressure–sinkage relations under a rolling/slipping wheel (with various slippage conditions), obtained by using our model, is displayed in Fig. 14. As can be seen, the pressure–sinkage relation changes in the entire contact area when the wheel slip changes. This overall behavior is consistent with the model recently proposed by Ding et al. [28] regarding slip–sinkage. In their approach, they modified the sinkage exponent $n$ as a linear function of slip ratio, in order to capture experimental observations. However, further experimental studies where stress distribution is measured at different slip ratios are necessary to fully investigate the validity of these models.

It is also noted that during elastic rebound, normal stress decreases linearly with $\zeta$, as a result of the linearly elastic relation used to represent the elastic behavior of soil. Of course, nonlinear elastic relations can also be incorporated to more accurately capture the elastic response of different types of soil.

Stress distributions at different slip ratios are displayed in Fig. 15 to better illustrate the situation. As can be seen from the figure, in the case of zero slip ratio, maximum normal stress is at the bottom-dead-center ($\theta = 0$); however, it shifts forward by increasing wheel slip. This shift in the location of maximum normal stress is caused by soil softening.\(^{11}\) For example, in the case of 20% or 40% slip ratio in Fig. 15, the normal stress increases from zero at $\theta = \theta_1$ to its maximum value at $\theta = \theta_M$. In this region, soil hardening is happening. From $\theta = \theta_M$ to around $\theta = 0$, soil softening happens and causes the normal stress to decrease. After that, normal and shear stresses decrease because of the elastic rebound of soil.

\(^{11}\)It should be noted that in the Drucker–Prager model used in this paper, when a plastic deformation happens at the failure surface $F_s$, or the transition surface $F_{tr}$, it may cause soil softening.

It should be mentioned that in Fig. 15, the location of the maximum shear stress is different from that of the normal stress. However, for a driven wheel with $i_s > 0$, i.e., 20% and 40% slip ratios, available experimental evidence, e.g., [38], suggests that the two points should be close to each other. Further experimental study and analysis with other constitutive relations for soil are required in this regard.

D. Discussion on the Proposed Velocity Field and Its Parameter Selection

To better illustrate the proposed velocity field, in Fig. 16, trajectories of soil particles initially at soil surface and 10 mm below surface are plotted at different slip ratios (wheel diameter is 0.3 m). The trajectories are overall consistent with reported experimental observations in [39]–[41]. The difference lies in the fact that soil particles initially located deeper in soil (not on the surface) tend to escape to soil surface where there is no loading (they move along the least resistive path); this behavior becomes more significant when soil compaction increases. What is important for our approach is the effect of this behavior on the velocity gradient (and strain tensor) in soil particles very close to the contact surface. Here, we have considered the effect of soil compaction by including $\varepsilon_{vol}^0$, which is the hardening/softening variable, in the velocity field.

In addition, in the proposed velocity field, it is assumed that a soil particle at the contact area has the same velocity as the adjacent point on the wheel surface. However, if the entry angle $\theta_1$ is large, as shown in Fig. 17, soil particles at the beginning of the contact may not follow the wheel surface motion. Furthermore, a soil particle at this location shows a higher tendency to escape to the surface. This means that its tendency to become compacted is lower. Even further, improvement to the model can be achieved by improving these aspects of the velocity field. This can be done based on experimental observations and analysis of the motion of soil particles under a wheel. For example, the soil visualization techniques used in [35] and [36] could be used for this.

Velocity field parameters, i.e., $\alpha_1$, $\alpha_2$, and $\alpha_3$, have to be identified in order to use the proposed model. However, these parameters are not inherent to either the Drucker–Prager...
parameters or the Bekker model (α₁ and α₃ have units of m⁻¹ and α₂ is dimensionless). These parameters, however, are related to soil properties. Based on our observation from various simulation trials, we noticed that by varying α₁ and α₂, we can cover a rather wide range of soil parameters, which are characterized by a range of values of n and k₀ of the Bekker model, without changing any parameter in the Drucker–Prager constitutive relation. Shear response can also be captured by tuning α₃ and φ (and c if soil cohesion is significant). This interesting behavior suggests that any parameter identification algorithm may need to focus on finding α₁ and α₂ based on the normal stress distribution, independent of α₃ and φ (and possibly c). This reduces the dimensionality and complexity of the identification problem. In addition, identifying the other Drucker–Prager parameters is often not needed.

Considering these observations, we used a trial-and-error approach in selecting velocity field parameters. To appropriately select α₁ and α₂, it is required to have normal stress distribution around the wheel at one operating condition with 10–20% wheel slippage. As this information is usually not available, we use the Bekker model to find a normal stress distribution. Here, we assume that the pressure–sinkage parameters are known. Parameters α₁ and α₂ are, then, selected such that the normal stress distribution (from the entry angle to the point of maximum stress) under the wheel at 20% slippage closely matches Bekker’s pressure–sinkage curve.

After selecting α₁ and α₂, α₃ is selected such that the traction force in an operating condition with 15–20% wheel slippage under a given vertical load on the wheel is close to the traction force calculated from the Bekker model. It should be noted that selection of these parameters is done at only one loading and operating condition, but the results are valid over a wide range of loading conditions, as shown in Figs. 4–6.

### E. Execution Time of the Proposed Algorithms

Plane-strain FE simulation of a rigid cylinder on soil with Abaqus/Explicit was conducted for comparison of execution time between the proposed approach and FEA. In this case, the wheel radius was 0.15 m and the wheel center velocity 0.00375 m/s with a slip ratio of 20%. The wheel was released with zero initial velocity and zero sinkage, while it was touching the soil surface. During the first 3.1 s, the wheel only moved in the vertical direction under gravity. Then, it was gradually accelerated to reach its final speed in 5 s. It continued moving with this velocity and 20% slip ratio. It took about 4 h to simulate 83.1 s of motion. The same motion was simulated using the new approach with a wheel using 72 mesh points, which resulted in eight to nine active mesh points, with total simulation time of about 180 s on the same processor (Intel Core 2 Duo T7500 at 2.2 GHz). This means about 80 times faster than with the FE simulation mentioned above. The code, based on the proposed approach, however, was written in MATLAB and only one single CPU core was used; therefore, a higher speedup is possible.

In addition, simulation of the Sojourner was conducted in Vortex using the proposed approach [16] on an Intel Core i7-920 processor at 2.67 GHz. This rover has six wheels, but all of the computations associated with the new model were performed on one single CPU core. During straight-line motion with a rover speed of 0.10 m/s, every second of motion took about 13-s simulation time, while the total number of active mesh points was 52 on average. This means about nine active mesh points per wheel. Since this simulation was performed using only one single CPU core for the computation associated with the new model, interactive and even real-time performance can
be achieved for slow-moving rovers if the data-parallel nature of this model is used. It should be noted that the computations associated with each active mesh point can be executed independently from the others, which makes the model data-parallel, and therefore, readily parallelizable. This feature, however, has not yet been implemented.

It is noted that for implementation on one single CPU core, the execution time depends linearly on the number of active mesh points. As the wheel sinkage grows, the number of active mesh points increases, which in turn can result in variation of the computational time. This is not desirable for real-time applications. However, parallel processing can help in improving that situation. The other variable that directly affects the execution time is the rover speed. A higher rover speed requires a smaller time-step to ensure the convergence of the elastoplasticity solver, which could, in turn, increase the overall execution time.

It is also noted that by decreasing the number of mesh points on the wheel, which in turn decreases the number of active mesh points, the discretization error increases. This could lead to substantial oscillatory behavior in the computed reaction forces. We recommend choosing the settings such that in a 2-D simulation of a wheel, at least five active mesh points cover the contact area.

F. Further Discussion

The results obtained with the proposed model and the assumed velocity field were close to those obtained based on the WR and Bekker models in the steady-state operating conditions for a particular slip ratio, as shown in this section.

Some of the limitations of the Bekker and WR models were listed in L1 and L2 (see Section II-A). To address limitation L1, a common remedy is adding a damping term in the z-direction as done by Azimi et al. [15] and by Sohl and Jain [17]. In [15], a variable damping term was added in the z-direction to include energy dissipation in this direction and damp any spurious oscillations. However, the damping coefficient was selected based on trial and error. The criterion used in setting the damping coefficient was to obtain a nonoscillatory response. However, in Algorithm 2 of this paper, the energy dissipation due to plastic deformation of soil is directly considered.

Regarding the slip–sinkage phenomenon L2, the results obtained with the proposed method showed a clear dependence of normal stress distribution on the slip ratio, with larger slip ratio resulting in larger wheel sinkage. This aspect was analyzed in Section IV-C. We have also compared the results of this approach with the experimental data reported in [42]; this comparison in Section V will show good agreement between the proposed model and the experimental data, regarding the slip–sinkage phenomenon.

Remark 10: A modification to the pressure–sinkage exponent $n$ in the WR model was suggested by Ding et al. [28], in order to capture the slip–sinkage behavior observed in their experimental data. To do so, they assumed that $n$ changes linearly with the slip ratio of the wheel, but no theoretical analysis was provided. □

V. Validation With Experimental Results

The experimental data reported by Ding et al. [42] are used for validation of the model proposed here. Wheel radius and width are 157.35 and 165 mm, respectively. The steady-state response of our model is compared with the experimental data for various values of slippage—see [42, Fig. 9(a)–(c)]. The experimental data reported for a wheel with no grousers, a wheel with 5-mm grousers, and a wheel with 10-mm grousers are used for comparison, as shown in Figs. 18–20. To this end, a single-wheel testbed is simulated in Vortex, in which forward and angular speeds of the wheel are controlled, while the wheel is free to move in the vertical direction, same as the example discussed in Section IV-B. The soil dataset used for parametrization of the elastoplastic constitutive relation is the same as what was used.
in the previous section, except for $\beta$ and $d$, which are set to $42.4^\circ$ and $367$ Pa, respectively, according to (24). This is done in order to comply with the soil properties reported in [42], where $\phi$ and $c$ are $31.9^\circ$ and $250$ Pa, respectively. It should be noted that the model parameters required for our elastoplastic constitutive relation were not available in [42].

In spite of that, by tuning the parameters of the velocity field according to Section IV-D, the terrain response is closely captured. Numerical values of the velocity field parameters are displayed in Table V.

The slip–sinkage phenomenon is captured by our model; however, as can be seen from Fig. 18, the sinkage is underestimated compared with the experimental data. In our model, the sinkage increases from almost $7$ mm at zero slip to around $15$ mm at $60\%$ slip, whereas in the experiments reported for a wheel with no grousers, the sinkage increases from around $6$ mm to around $17.5$ mm.

Even though the predicted wheel sinkage values are not exactly the same as in the experimental data, it is noteworthy that generally, the experimentally observed behavior is naturally captured by the proposed model. In addition, the drawbar pull and driving torque estimation match the experimental data fairly well, as shown in Figs. 19 and 20.

In order to further verify the scalability of our model, its prediction is compared with the experimental data under the different vertical loads reported in [42]. However, the only available data are for a wheel with $10$-mm grousers under $35$-, $80$-, and $150$-N vertical loads. The comparison is displayed in Figs. 21–23. As expected, the slip–sinkage is underestimated, due to the extra sinkage that these relatively big grousers have caused in the experiments. Drawbar pull and driving torque (see Figs. 22 and 23, respectively), are captured relatively well.

VI. CONCLUSION

We have introduced a novel approach to wheel–soil interaction modeling that allows for efficient dynamics modeling, simulation and analysis of rovers, and provides a theoretical framework in wheel–soil interaction. In the proposed approach, instead of using FEA to find the stress field in the contact area,
we have proposed a velocity field for soil particles in the vicinity of the contact region. Using this velocity field and the modified Drucker–Prager constitutive relation for soil, the stress field in the contact area is determined incrementally using an explicit integration scheme explained in Algorithm 2. In this approach, soil plastic deformation and energy loss due to this deformation is captured. In addition, elastic rebound of soil is determined using Algorithm 3. The proposed approach is modular in that it does not have to be used with the Drucker–Prager constitutive relation and the proposed velocity field. Any elastoplastic constitutive relation and other possible velocity field representations can be used in this framework, if needed, to more closely capture the behavior of different soil types.

The results are in good agreement with experimental data available in the literature and also with results that can be obtained based on the Bekker and WR models. In addition, the proposed model is fully compatible with multibody dynamics environments. Energy dissipation due to soil plastic deformation is directly represented in our model. Furthermore, the slip–sinkage phenomenon is captured by the model as a natural outcome.

The processing time required for the computation of soil response using the proposed model is much shorter than for FEA, and its implementation is readily parallelizable. Therefore, the framework introduced here allows for fast and interactive simulation of rovers on soft soil, while a high-fidelity elastoplastic constitutive relation can be used for soil. In addition, the model has the capability of covering a broad range of motion possibilities that can happen under general dynamic motion conditions.

**APPENDIX A**

**DRUCKER–PRAGER WITH CAP HARDENING**

Yield surfaces and flow potentials are defined in terms of the stress invariants $p$ and $q$; $p$ is related to the first moment of the stress tensor

$$p = -\frac{1}{3} \text{tr}(\sigma)$$

while $q$, i.e., the von Mises equivalent stress, is proportional to the square root of the second moment of $\mathbf{S}$:

$$q = \sqrt{\frac{3}{2} \text{tr}(\mathbf{S}^2)}$$

where $\mathbf{S}$ is the stress deviator, which is defined as

$$\mathbf{S} = \mathbf{\sigma} + \mathbf{pI}$$

and $\mathbf{I}$ is the identity tensor.

1) *Yield Surface*: The yield surface $F_s$ as depicted in Fig. 24, consists of three parts in the $pq$ plane: the failure surface $F_s$, the transition surface $F_c$, and the cap surface $F_c$. They obey the following relations [25]:

$$F_s = q - p \tan \beta - d = 0$$

$$F_c = \sqrt{(p - p_a)^2 + \left( \frac{R_c q}{1 + \alpha - \alpha / \cos \beta} \right)^2} - R_c (d + p_a \tan \beta) = 0$$

where $\beta$ and $d$ are related to the angle of friction and cohesion of the material, respectively. As mentioned in [25], $R_c$ is a material parameter referred to as the cap eccentricity, while $\alpha$, which is typically between 0.01 and 0.05, is used to define the size of $F_{tr}$, and $p_a$ is an evolution parameter that defines the hardening or softening behavior. The hardening/softening relation is a piecewise linear function between $\varepsilon_p^s$ and $p_b$, with $\varepsilon_p^s$ denoting the volumetric plastic strain and $p_b$ the hydrostatic compression yield stress. In this constitutive model, $p_a$ is computed from the relation [25]

$$p_a = \frac{p_b - R_c d}{1 + R_c \tan \beta}.$$  

2) *Flow Potential Surface*: Plastic flow is defined by a flow potential surface $G$ that consists of two parts in the $pq$ plane: plastic flow on the cap region $G_c$ and plastic flow on the failure and transition regions $G_s$ [25]

$$G_c = \sqrt{(p - p_a)^2 + \left( \frac{R_c q}{1 + \alpha - \alpha / \cos \beta} \right)^2}$$

$$G_s = \sqrt{[(p_a - p) \tan \beta]^2 + \left( \frac{q}{1 + \alpha - \alpha / \cos \beta} \right)^2}.$$

The two elliptical portions $G_c$ and $G_s$ form a $C^1$-continuous potential surface.

**APPENDIX B**

**COMPUTATION OF THE ELASTIC AND PLASTIC PARTS OF THE STRAIN INCREMENT TENSOR**

The calculation of $\Delta \varepsilon^e$ and $\Delta \varepsilon^p$ is explained briefly here, based on the total strain increment $\Delta \varepsilon$.

Following an approach explained in [33], the elastic and plastic parts of the strain increment tensor can be decomposed by finding a single scalar, known as the plastic multiplier, from a
nonlinear algebraic equation. This procedure is summarized as follows.

Using plastic potential theory, a classical plasticity theory [34], the increment in the plastic strain can be obtained as

$$\Delta \varepsilon^p = \lambda \frac{\partial G}{\partial \sigma}$$  \hspace{1cm} (34)

where $\lambda$ is the plastic multiplier, and $G$ is the flow potential surface defined in (32) and (33) for the Drucker–Prager constitutive relation.

In the algorithm presented below, it is required to find the evolution of $p_b$, i.e., the hardening variable in the Drucker–Prager constitutive relation, based on $\lambda$. As $p_b$ depends only on $\varepsilon_{\text{vol}}^p$, the following relation for the evolution of $p_b$ is derived using (34):

$$\Delta p_b = \lambda \frac{\partial p_b}{\partial \varepsilon_{\text{vol}}^p} \text{trace} \left( \frac{\partial G}{\partial \sigma} \right).$$  \hspace{1cm} (35)

In obtaining (35), the relation for the volumetric plastic strain, namely,

$$\varepsilon_{\text{vol}}^p = \text{trace}(\varepsilon^p)$$  \hspace{1cm} (36)

is used.

For determining $\lambda$, what is called the loading/unloading conditions of the elastoplastic model must be satisfied [33]. These conditions can be expressed in terms of the Karush–Kuhn–Tucker criteria [43]:

$$F(\sigma, p_b) \leq 0; \quad \lambda \geq 0; \quad F \cdot \lambda = 0$$  \hspace{1cm} (37)

where $F$ is the yield surface that is defined for the Drucker–Prager constitutive relation in (28)–(30). Based on these criteria, a positive $\lambda$ exists when there is a plastic deformation; $\lambda$ is zero if the deformation is only elastic. Using the condition stated in (37), an elastic predictor/plastic corrector algorithm is normally used to compute $\lambda$ and, thus, the updated stress values. This algorithm involves the steps below (following the procedure explained in [33, Box 7.1]):

Algorithm 4.

Given the current stress tensor $\sigma(0)$, strain tensors $\varepsilon^e(0)$ and $\varepsilon^p(0)$, and the total strain increment $\Delta \varepsilon$, find $\lambda$.

a) Elastic predictor step (adapted from [33]):

i) Set $\lambda = 0$, and calculate the elastic trial states:

$$\varepsilon_{\text{trial}}^e = \varepsilon^e + \Delta \varepsilon$$  \hspace{1cm} (38a)

$$p_{\text{trial}}^b = p_b(0)$$  \hspace{1cm} (38b)

$$\sigma_{\text{trial}} = C : \varepsilon_{\text{trial}}^e$$  \hspace{1cm} (38c)

It should be noted that $p_b$ changes with $\varepsilon_{\text{vol}}^p$. Therefore, if $\varepsilon_{\text{vol}}^p(0)$ is known, so is $p_b(0)$.

ii) Verify the plastic admissibility condition: $F(\sigma_{\text{trial}}, p_b) \leq 0$. If the check fails, we go to step (b); otherwise, $\lambda$ is reported as zero, i.e., only elastic deformation occurs.

b) Plastic corrector step or Return-Mapping Algorithm, in which the system of equations below is solved for $\varepsilon^e$, $p_b$, and $\lambda$ (adapted from [33]):

$$\varepsilon^e - \varepsilon_{\text{trial}}^e + \lambda \frac{\partial G}{\partial \sigma} = 0$$

$$p_b - p_{\text{trial}}^b - \lambda \frac{\partial p_b}{\partial \varepsilon_{\text{vol}}^p} \text{trace} \left( \frac{\partial G}{\partial \sigma} \right) = 0$$

$$F(\sigma, p_b) = 0$$  \hspace{1cm} (39)

with $\sigma = C : \varepsilon^e$. More details on numerical solutions of (39) can be found in [43].

\section*{Appendix C}

\textbf{Plate-Penetration Test With Abaqus}

As mentioned in Section IV, we simulated the plate-penetration test (bevameter test) using Abaqus to identify $k_c$, $k_\phi$, and $n$. For this, 2-D simulation runs were conducted on Abaqus/Explicit with plate widths of 100 and 120 mm. The von Mises stress distribution under the plate is shown in Fig. 25. It should be noted that only one-half of the plate and soil mass is simulated, as the problem has a plane of symmetry. The average pressure developed under the plate is plotted in Fig. 26 versus plate sinkage for both plate sizes. From that figure, $k_c$, $k_\phi$, and $n$ are identified.
ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their comments.

REFERENCES


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